Exactly solvable potentials

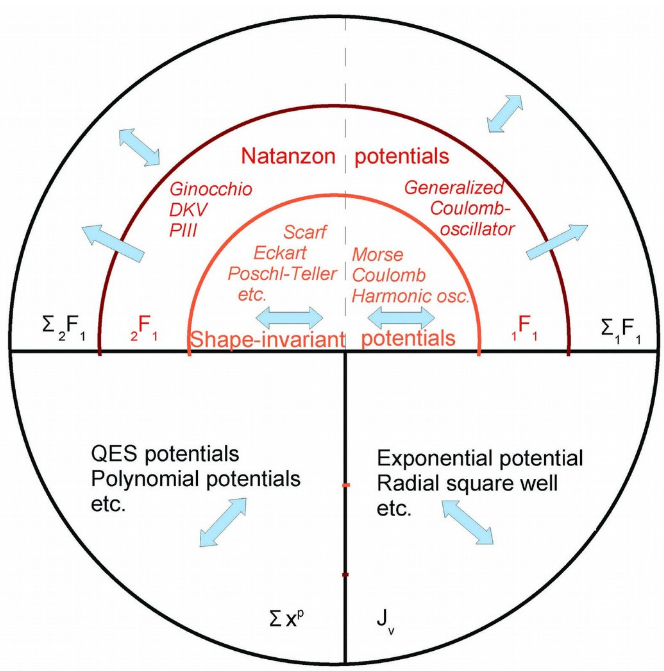
Local potentials have been used to model the interactions of the subatomic world ever since the introduction of quantum mechanics. Some of these (like the Coulomb potential) do not differ essentially from the forces observed in nature, while most of them (like the harmonic oscillator, for example) represent approximations of the actual physical situation. The potential shape, defined by the potential type and the parameters in it is usually chosen in a way that reflects the physical picture our intuition associates with the problem; therefore we can define attractive or repulsive, short-range or long-range potentials, etc. The concept of potentials is deeply rooted in the thinking of most physicist. This is perhaps not surprising, because the most elementary examples introduced at the dawn of quantum mechanics still form essential part of any quantum mechanical course, and also play a fundamental role in the formulation of most physical models of the microscopic world.

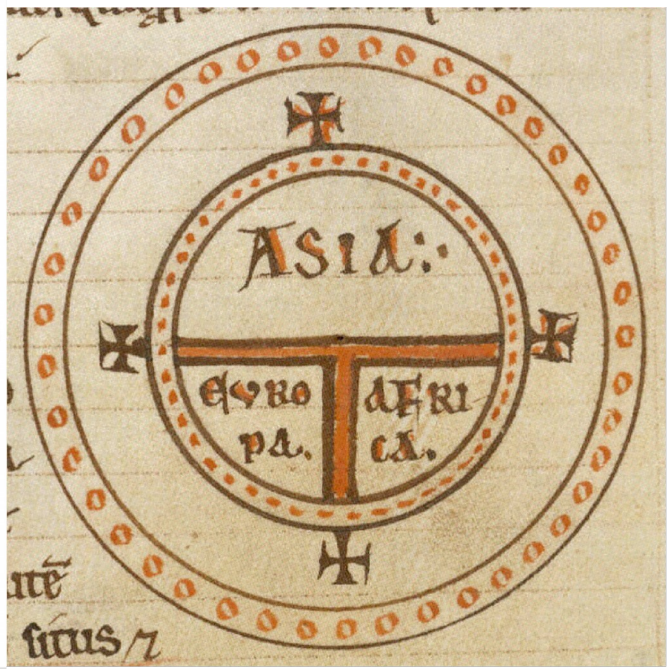
Some of the potentials used in quantum mechanics are exactly solvable. This means that the energy eigenvalues, the bound-state wave functions and the scattering matrix can be determined in closed analytical form. The range of these potentials has been extended considerably in the recent years by investigations inspired by some novel symmetry-based approaches. The concept of solvability has also been extended: one can talk about conditionally exactly or quasi-exactly solvable potentials too, in addition to the "classical" exactly solvable examples. Due to these developments more and more interactions can be modeled by making advantage of the increasingly flexible potential shapes offered by solvable potentials. Their solutions can be applied directly, or can be combined with numerical calculations. In the simplest case analytical calculations can aid numerical studies in areas where numerical techniques might not be safely controlled. This is the case, for example, when bound-state wave functions with arbitrary node numbers are required, for certain singular potentials, complex potentials, or in situations, when the physical system crosses a critical point. As the next level of complexity, analytical solutions can supply a basis for numerical calculations. This makes exactly solvable problems indispensable even in the age of rapidly developing computational resources. Besides their role in describing realistic physical problems, solvable quantum mechanical potentials also represent an interesting field of investigation in their own right. This is largely due to the mathematical elegance and beauty associated with the symmetries of these problems.

The exact description of quantum mechanical potential problems is usually performed by transforming the Schrödinger equation into the second-order differential equation of some special function of mathematical physics. Depending on the choice of this function and on the variable transformation, various classes of exactly solvable potentials can be obtained. The most widely discussed potentials belong to the six-parameter Natanzon class, in which case the solutions of the Schrödinger equation are obtained in terms of a single hypergeometric or confluent hypergeometric function. For bound states these special functions reduce to Jacobi and generalized Laguerre polynomials, respectively. The most widely known exactly solvable potentials (harmonic oscillator, Coulomb, Pöschl-Teller, etc.) are two- or three-parameter Natanzon-class potentials, and belong to the so-called shape-invariant potential class.

Supersymmetric quantum mechanics (SUSYQM) is another standard tool of analysing exactly solvable potentials. This method can be used to generate new exactly solvable potentials from known ones such that the two potentials are isospectral, except perhaps for their ground states. The two potentials are called supersymmetric partners and their degenerate levels are connected by a first-order differential operator. The solutions of the new potential generally contain two terms: one with the special function appearing in the solution of the original potential and one with its first-order derivative, so the new potential of usually outside the Natanzon-class. However, in special situations the two terms can be reduced to only one by applying recurrence relations, so the new potential will have the same mathematical structure, and the two potentials will differ only in the parameters appearing in them. In this case the potential belongs to the shape-invariant class mentioned earlier.

There are several ways to obtain exactly solvable potentials beyond the Natanzon class. One is considering further special functions of mathematical physics (e.g. the Bessel function, Heun functions, etc.), while another one is considering the linear combination of several special functions of the same type in the solutions.





A comprehensive review on generating and classifying exactly solvable potentials both in the conventional and in the PT-symmetric setting can be found in Chapter 7 of *"PT Symmetry in Quantum and Classical Physics"* by C. M. Bender et al., World Scientific Europe Ltd., London, 2019.

Studies focusing on conventional (hermitian) quantum mechanics

Our activity in this field mainly concerned the exact solution of the one-dimensional Schrödinger equation. We gave a systematic treatment of shape-invariant potentials [1], which contain the most well-known textbook examples for solvable potentials (Coulomb, harmonic oscillator, Morse, Pöschl-Teller, Scarf, Rosen-Morse, Eckart) and introduced a straightforward classification scheme for them

[1,2]. We extended the same method to describe several three- and four-parameter members of the more general six-parameter Natanzon potential class [3-8].

We also applied the techniques of supersymmetric quantum mechanics (SUSYQM) to discuss shape-invariant [9,24], Natanzon-class [10] and conditionally solvable potentials [11], extending the investigations also to complex potentials [10,12]. The relation of SUSYQM and su(1,1) potential and spectrum generating algebras has also been explored for shape-invariant potentials [13]. We also applied a semi-numerical method (the asymptotic iteration method) to obtain the bound-state energy eigenvalues of some quasi-exactly solvable potentials and their supersymmetric partner [22].

We extended our analysis to various types of potentials beyond the Natanzon class. We considered the bi-confluent Heun functions to generate the solutions of the sextic oscillator, which also has the property of being quasi-exactly solvable (QES). This means that the solutions of the lowest few eigenstates can be obtained in closed form, for certain constraints of the potential parameters. We discussed how and when the QES solutions can be obtained from the alternative treatment [25,26].

As another example for potentials beyond the Natanzon class, we discussed in a pedagogical work SUSYQM methods to obtain the rational extension of the radial harmonic oscillator, the solutions of which are written in terms of X1-type exceptional Laguerre polynomials [27].These potentials are expressed in terms of two conventional generalized Laguerre polynomials, indicating that the corresponding potentials are, indeed, outside the Natanzon class.

The radial version of the Scarf II potential is another example for such potentials, because in this case the solutions are expressed in terms of both independent solutions of the hypergeometric differential equation, which have to be matched at the origin [28]. (This mechanism is the same as the one from which the Woods-Saxon potential can be obtained from the Rosen-Morse II potential.) We investigated the dependence of the S-matrix poles on the choice of the parameters.

Our further activity concerned developing Green's operator techniques [14,15] for the Coulomb potential and one of its generalizations [6], combining the orbital structure of solvable potentials with spin degrees of freedom [16], as well as deriving some exact solutions of the Dirac equation [17]. As a further result we extended our investigation to Schrödinger equations with position-dependent mass (PDM) and introduced a method by which exactly solvable PDM problems can be generated with mass functions that are finite and non-zero everywhere [18]. Inspired by some of our results concerning the PT-symmetric Coulomb potential, we investigated the conditions under which the M(x) function can take on negative values [23].

We used the quasi-exactly solvable sextic oscillator to describe shape phase transitions of certain nuclei within the Bohr Hamiltonian [19]. By fitting the energy spectrum we described several nuclei that are near the phase transition that takes place between the spherical and gamma-unstable shape phases [21]. We formulated a parameter-free condition for the location of this phase transition [20].

Thematical index

Items denoted by PTnn refer to publications on PT-symmetric quantum mechanics in the ensuing section

Exactly solvable potentials
- Shape-invariant potentials [1, 2, 9, 12, 13, 24]
- Natanzon-class potentials [3, 4, 5, 6, 7, 8, 9, 10, 18, PT26]

- Potentials beyond the Natanzon class [25, 26, 27, 28]

Conditionally exactly solvable potentials [11]

Quasi-exactly solvable potentials [19, 20, 21, 22]

Exact expressions for normalization constants [PT8, PT10, PT11]

SUSYQM [9, 10, 11, 12, 22, 24]

Algebraic aspects [6, 13]

Spin-dependent and relativistic problems [16, 17]

Green's operator techniques [14, 15]

Problems with position-dependent effective mass [18, 23]

Applications in nuclear physics [19, 20, 21]

[1] G. Lévai:
"A search for shape-invariant solvable potentials"
Journal of Physics A 22 (1989) 689

[2] G. Lévai:
"On some exactly solvable potentials derived from supersymmetric quantum mechanics"
Journal of Physics A 25 (1992) L521

[3] G. Lévai:
"A class of exactly solvable potentials related to the Jacobi polynomials"
Journal of Physics A 24 (1991) 131

[4] G. Lévai, B. W. Williams:
"The generalized Coulomb problem: an exactly solvable model"
Journal of Physics A 26 (1993) 3301

[5] B. W. Williams, J. L. Rutherford, G. Lévai:
"'Implicit' potentials associated with Jacobi polynomials: some novel aspects"
Physics Letters A 199 (1995) 7

[6] G. Lévai, B. Kónya, Z. Papp:
"Unified treatment of the Coulomb and harmonic oscillator potentials in D dimensions"
Journal of Mathematical Physics 39 (1998) 5811

[7] R. Roychoudhury, P. Roy, M. Znojil, G. Lévai:
"Comprehensive analysis of conditionally exactly solvable models"
Journal of Mathematical Physics 42 (2001) 1996

[8] B. W. Williams, G. Lévai:
"An asymmetric "implicit" potential on the real line'"
Modern Physics Letters A 18 (2003) 1901

[9] G. Lévai:
"Solvable potentials derived from supersymmetrical quantum mechanics"
Lecture Notes in Physics 427 (1994) 107

[10] G. Lévai, D. Baye, J.-M. Sparenberg:
"Phase-equivalent potentials from supersymmetry: analytical results for a Natanzon-class potential"
Journal of Physics A 30 (1997) 8257

[11] G. Lévai, P. Roy:
"Conditionally exactly solvable potentials and supersymmetric transformations"
Physics Letters A 264 (1999) 117

[12] D. Baye, G. Lévai, J.-M. Sparenberg:
"Phase-equivalent complex potentials"
Nuclear Physics A 599 (1996) 435

[13] G. Lévai:
"Solvable potentials associated with su(1,1) algebras: a systematic study"
Journal of Physics A 27 (1994) 3809

[14] B. Kónya, G. Lévai, Z. Papp:
"Green's matrix from Jacobi-matrix Hamiltonian"
Journal of Mathematical Physics 38 (1997) 4832

[15] B. Kónya, G. Lévai, Z. Papp:
"Continued fraction representation of the Coulomb Green's operator and unified description of bound, resonant and scattering states"
Physical Review C 61 (2000) 3:4302

[16] G. Lévai, F. Cannata:
"Factorization of spin-dependent Hamiltonians"
Journal of Physics A 32 (1999) 3947

[17] G. Lévai, A. Del Sol:

"On some solutions of the Dirac equation"
Journal of Physics A 29 (1996) 2827

[18] G. Lévai, O. Özer:
"An exactly solvable Schrödinger equation with finite positive position-dependent effective mass"
Journal of Mathematical Physics 51 (2010) 9:2103

[19] G. Lévai, Arias J. M.:
"The sextic oscillator as a gamma-independent potential"
Physical Review C 69 (2004) 1:4304

[20] G. Lévai, Arias J. M.:
"Search for critical-point nuclei in terms of the sextic oscillator"
Physical Review C 81 (2010) 4:4304

[21] G. Lévai:
"*The sextic oscillator and E(5) type nuclei"*
Journal of Physics: Conference Series 205 (2010) 1:2023

[22] O. Özer, G. Lévai:
"*Asymptotic iteration method applied to bound-state problems with unbroken and broken supersymmetry"*
Romanian Journal of Physics 57 (2012) 582

[23] M. Znojil, G. Lévai:
"Schrödinger equations with indefinite effective mass"
Physics Letters A 376 (2012) 3000

[24] G. Lévai, D. Baye, J.-M. Sparenberg:
"*Analytical results on generating phase-equivalent potentials by supersymmetry: Removal and addition of bound states"*
Lecture Notes in Physics 488 (1997) 363

[25] G. Lévai, A. M. Ishkhanyan:
"Exact solutions of the sextic oscillator from the bi-confluent Heun equation"
Modern Physics Letters A 34 (2019) 1950134

[26] A. M. Ishkhanyan, G. Lévai:
"Hermite function solutions of the Schrödinger equation for the sextic oscillator"
Physica Scripta 95 (2020) 085202

[27] G. Lévai, T. Soltész:
"Unified supersymmetric transformations for the harmonic oscillator and its rational extension"
European Journal of Physics 41 (2020) 025403

[28] G. Lévai, Á. Baran, P. Salamon, T. Vertse:
"Analytical solutions for the radial Scarf II potential'"
Physics Letters A 381 (2017) 1936-1942

Studies focusing on PT-symmetric quantum mechanics

PT-symmetric quantum mechanical systems are invariant under the simultaneous action of the P space and T time inversion operations. These systems possess non-hermitian Hamiltonians, still they have some characteristics similar to hermitian problems. The most notable of these is their discrete energy spectrum, which can be partly or completely real. Typically the transition from the fully real energy spectrum to the complex one occurs when the non-hermitian component of the Hamiltonian exceeds a certain critical limit, and it can be interpreted as the spontaneous breakdown of PT symmetry in that the energy eigenstates cease to be eigenstates of the PT operator then.

Another typical feature PT-symmetric systems have in common with hermitian problems is that their basis states form an orthogonal set with a redefined inner product as $\langle\Psi|\Phi\rangle_{PT} = \langle\Psi|P\ \Phi\rangle$. Similarly to the hermitian setting, the so-called pseudo-norm defined by this inner product is conserved, however, a major difference is that it turned out to possess indefinite sign, and this raised the question of the probabilistic interpretation of PT-symmetric systems. PT symmetry also manifests itself in scattering aspects in that the reflection coefficient exhibits handedness.

Perhaps the simplest PT-symmetric Hamiltonian contains a one-dimensional Schrödinger operator with a complex potential satisfying the $V^*(-x)=V(x)$ relation. A number of such problems have been described by numerical and perturbational techniques, but the exact analytical solution of several potentials have also been given.

PT symmetry was put into a more general context when it was found that it is a special case of pseudo-hermiticity, and this explains most of the peculiar features of PT-symmetric systems. It was shown that PT-symmetric, and in general, pseudo-hermitian systems can be mapped into equivalent hermitian ones, although this mapping is technically not straightforward in general. A significant result was the experimental verification of the existence of PT-symmetric systems which also exhibit the spontaneous breakdown of PT symmetry [C. E. Rüter et al., Nature Physics 6 (2010) 192]. One can say that PT-symmetric quantum mechanics originated as a curiosity in mathematical physics, but it took only slightly more than a decade to reach a stage in which its practical applications seem possible.

Our first results concerning PT-symetric potentials was the systematic exploration of conditions under which shape-invariant potentials possess real [1] and complex [2] energy spectrum. These studies revealed that the PT-symmetric Coulomb potential cannot be defined on the real x axis, rather one has to define an integration path in the complex x plane [3]. This result also raised several further questions concerning the definition of the PT-symmetric Coulomb potential, which have been settled later both for bound [4] and scattering states [5].

In a series of papers we discussed various aspects of PT-symmetric shape-invariant potentials, including the conditions for the occurrence of the spontaneous breakdown of PT-symmetry [8,9,13,26], the explicit expression for the pseudo-norm [8,10,11,12] and the transmission and reflection coefficients [6,7,12]. We discussed the asymptotic properties of PT-symmetric potentials and pointed out that a dominant (i.e. asymptotically non-vanishing) imaginary potential component may lead to unusual features (e.g. the cancellation of complex energy eigenvalues and the spontaneous breakdown of PT symmetry, handedness of the T(k) transmission coefficient) [25].

We extended our analysis to wider classes of solvable potentials. In particular, we considered some members of the Natanzon potential class [14,15,26] We constructed a three-parameter potential [26] that contains all the shape-invariant potentials as a special limit, as well as the Dutt-Khare-Varshni potential [15]. We showed that the breakdown of PT symmetry, i.e. the complexification of the energy eigenvalues occurs in this case through the gradual mechanism, starting from the ground state. We also explored how the PT-symmetric version of the general Natanzon potentials can be introduced [27]. This study also revealed why the breakdown of PT symmetry cannot occur for the PII type shape-invariant potentials, i.e. for the Rosen-Morse I, II and Eckart potentials. We demonstrated that the accidental crossing of energy levels, which was found previously for a few PT-symmetric potentials is a general feature of those PT-symmetric Natanzon-class potentials, for which the q quasi-parity quantum number can be defined [28].

We also studied PT-symmetric conditionally exactly solvable [16] potentials, as well as exactly non-solvable potentials [17]. In the latter case we presented a thorough analysis of the finite PT-symmetric square well potential [29]. We found that this potential can support only states with real energy eigenvalues, similarly to the Rosen-Morse II potential, and we attributed this finding to their asymptotically non-vanishing imaginary potential component. We also discussed the special common limit of these two potentials, i.e. the one containing a real Dirac delta and an imaginary step function [30].

Based on the results for one-dimensional exactly solvable PT-symmetric potentials, we also discussed such potentials in two and three spatial dimensions, which can be factorized into one-dimensional problems by means of separating the radial and angular variables [18-20].

We also discussed the relation of PT symmetry with Lie algebraic and supersymmetric formulations of non-relativistic potential problems [21]. In particular, we pointed out the importance of the so(2,1) [6] and so(2,2) [7] algebras as potential algebras for the Scarf II and the Pöschl-Teller potentials. We demonstrated that while for unbroken PT symmetry the Scarf II potential has two distinct SUSYQM partners, the spontaneous breakdown of PT symmetry results in the manifest breakdown of supersymmetry for this system [22]. We also derived general conditions for the simultaneous occurrence of PT symmetry and supersymmetry in quantum mechanical potential problems [23].

Thematical index

Shape-invariant potentials
- in general [1, 2]

- Coulomb [3, 4, 13, 25]
- Scarf II [6, 7, 8, 9, 22, 24, 25]
- Rosen-Morse I [11, 23]

- Rosen-Morse II [12, 25]
- Generalized Pöschl-Teller [7]

Natanzon-class potentials [27,28]

Conditionally exactly solvable potentials (beyond the Natanzon class) [16]

Exactly non-solvable potentials [17,29,30]

Exact expressions for the pseudo-norm and normalization constants [8,10,11,12

Spontaneous breakdown of PT symmetry

- Sudden mechanism [2, 8, 9, 13, 14, 19, 22, 24]
- Gradual mechanism [15, 17, 26]

- Does not happen [10, 11, 12, 29, 30]

SUSYQM [21, 22, 23]
Algebraic aspects [6, 7, 21]

[1] G. Lévai, M. Znojil:
"Systematic search for PT-symmetric potentials with real energy spectra"
Journal of Physics A 33 (2000) 7165

[2] G. Lévai, M. Znojil:
"Conditions for complex spectra in a class of PT symmetric potentials"
Modern Physics Letters A 16 (2001) 1973

[3] M. Znojil, G. Lévai:
"The Coulomb - harmonic oscillator correspondence in PT symmetric quantum mechanics"
Physics Letters A 271 (2000) 327

[4] M. Znojil, P. Siegl, G. Lévai:
"Asymptotically vanishing PT-symmetric potentials and negative-mass Schrödinger equations"
Physics Letters A 373 (2009) 1921

[5] G. Lévai, P. Siegl P., M. Znojil:
"Scattering in the PT-symmetric Coulomb potential"
Journal of Physics A 42 (2009) 29:5201

[6] G. Lévai, F. Cannata, A. Ventura:
"Algebraic and scattering aspects of a PT-symmetric solvable potential"
Journal of Physics A 34 (2001) 839

[7] G. Lévai, F. Cannata, A. Ventura:
"PT-symmetric potentials and the so(2,2) algebra"
Journal of Physics A 35 (2002) 5041

[8] G. Lévai, F. Cannata, A. Ventura:
"PT symmetry breaking and explicit expressions for the pseudo-norm in the Scarf II potential"
Physics Letters A 300 (2002) 271

[9] G. Lévai:
"Exact analytic study of the PT-symmetry-breaking mechanism"
Czechoslovak Journal of Physics 54 (2004) 77

[10] G. Lévai:
"On the pseudo-norm and admissible solutions of the PT-symmetric Scarf I potential"
Journal of Physics A 39 (2006) 10161

[11] G. Lévai:
"On the normalization constant of PT-symmetric and real Rosen-Morse I potentials"
Physics Letters A 372 (2008) 6484

[12] G. Lévai, E. Magyari:
"The PT-symmetric Rosen-Morse II potential: effects of the asymptotically non-vanishing imaginary potential component"
Journal of Physics A 42 (2009) 19:5302

[13] G. Lévai:
"Spontaneous breakdown of PT symmetry in the complex Coulomb potential"
Pramana - Journal of Physics 73 (2009) 2:329

[14] G. Lévai, A. Sinha, P. Roy:
"An exactly solvabte PT symmetric potential from the Natanzon class"
Journal of Physics A 36 (2003) 7611

[15] M. Znojil, G. Lévai, P. Roy, R. Roychoudhury:
"Anomalous doublets of states in a PT symmetric quantum model"
Physics Letters A 290 (2001) 249

[16] A. Sinha, G. Lévai, P. Roy:
"PT symmetry of a conditionally exactly solvable potenital"
Physics Letters A 322 (2004) 78

[17] M. Znojil, G. Lévai:
"Spontaneous breakdown of PT symmetry in the solvable square-well model"
Modern Physics Letters A 16 (2001) 2273

[18] G. Lévai:
"Solvable PT-symmetric potentials in higher dimensions"

Journal of Physics A 40 (2007) F273

[19] G. Lévai:
"Solvable PT-symmetric potentials in higher dimensions"
Journal of Physics A 41 (2008) 24:4015

[20] G. Lévai:
"Solvable PT-symmetric potentials in 2 and 3 dimensions"
Journal of Physics: Conference Series 128 (2008) 1:2045(12)

[21] G. Lévai:
"SUSYQM and other symmetries in quantum mechanics"
Journal of Physics A 37 (2004) 10179

[22] G. Lévai, M. Znojil:
"The interplay of supersymmetry and PT symmetry in quantum mechanics: a case study for the Scarf II potential"
Journal of Physics A 35 (2002) 8793

[23] G. Lévai:
"Symmetries without hermiticity"
Czechoslovak Journal of Physics 54 (2004) 1121

[24] G. Lévai:
"Comparative analysis of real and PT-symmetric Scarf potentials"
Czechoslovak Journal of Physics 56 (2006) 953

[24] G. Lévai:
"*Comparative analysis of real and PT-symmetric Scarf potentials"*
Czechoslovak Journal of Physics 56 (2006) 953

[25] G. Lévai:
"*Asymptotic properties of solvable PT-symmetric potentials"*
International Journal of Theoretical Physics 50 (2011) 4:997

[26] G. Lévai:
"Gradual spontaneous breakdown of PT symmetry in a solvable potential"
Journal of Physics A 45 (2012) 44:4020

[27] G. Lévai:
"*PT Symmetry in Natanzon-class Potentials'"*
International Journal of Theoretical Physics 54 (2015) 2724-2736

[28] G. Lévai:
"*Accidental crossing of energy levels in PT-symmetric Natanzon-class potentials"*
Annals of Physics 380 (2017) 1-11

[29] G. Lévai, J. Kovács:
"The finite PT-symmetric square well potential"
Journal of Physics A 52 (2019) 025302

[30] J. Kovács, G. Lévai:
"On the common limit of the PT-symmetric rosen–morse II and finite square well potentials"
Acta Polytechnica 57 (2017) 412-417